

VARIATIONAL CALCULUS ON WIENER SPACE WITH RESPECT TO CONDITIONAL EXPECTATIONS

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Abstract: We give a variational formulation for $-\log \mathbb{E}_\nu [e^{-f} | \mathcal{F}_t]$ for a large class of measures ν . We give a refined entropic characterization of the invertibility of some perturbations of the identity. We also discuss the attainability of the infimum in the variational formulation and obtain a Prékopa-Leindler theorem for conditional expectations.

Keywords: Wiener space, variational formulation, entropy, invertibility, Brownian bridge, loop measure, diffusing particles, conditional expectation, Prékopa-Leindler theorem

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1. Introduction

Denote \mathbb{W} the space of continuous functions from $[0, 1]$ to \mathbb{R}^n and \mathbb{H} the associated canonical Cameron-Martin space of elements of \mathbb{W} which admit a density in L^2 . Also denote μ the Wiener measure, W the coordinate process, and (\mathcal{F}_t) the canonical filtration of \mathbb{W} completed with respect to μ . W is a Brownian motion under μ . Set f a bounded from above measurable function from \mathbb{W}

to \mathbb{R} . In [5], Dupuis and Ellis prove that

$$(1.1) \quad -\log \mathbb{E}_\mu [e^{-f}] = \inf_{\theta} (\mathbb{E}_\theta [f] + H(\theta|\mu))$$

where the infimum is taken over the probability measures θ on \mathbb{W} which are absolutely continuous with respect to μ and the relative entropy $H(\theta|\mu)$ is equal to $\mathbb{E}_\mu \left[\frac{d\theta}{d\mu} \log \frac{d\theta}{d\mu} \right]$. In [1], Boué and Dupuis use it to derive the variational formulation

$$(1.2) \quad -\log \mathbb{E}_\mu [e^{-f}] = \inf_u \mathbb{E}_\mu \left[f \circ (W + u) + \frac{1}{2} \int_0^1 |\dot{u}(s)|^2 ds \right]$$

where the infimum is taken over L^2 functions from \mathbb{W} to \mathbb{H} whose density is adapted to (\mathcal{F}_t) . This variational formulation is useful to derive large deviation asymptotics as Laplace principles for small noise diffusions for instance. This result was later extended by Budhiraja and Dupuis to Hilbert-space-valued Brownian motions in [2], and then by Zhang to abstract Wiener space in [21], using the framework developed by Üstünel and Zakai in [18].

The Prékopa-Leindler theorem first formulation was given by Prékopa in [14] and arose in stochastic programming where a lot of non-linear optimization problems require concavity. In [8], Huu Hariya uses the variational formulation to retrieve a Prékopa-Leindler theorem for Wiener space, similar to the formulation of Üstünel in [7] with log-concave measures. Other functional inequalities can be derived from 1.2, see for instance Lehec in [12].

The bounded from above hypothesis in 1.2 was weakened significantly by Üstünel in [20], it was replaced with the condition

$$\mathbb{E}_\mu [f e^{-f}] < \infty$$

and the existence of conjugate integers p and q such that

$$f \in L^p(\mu), e^{-f} \in L^q(\mu)$$

These relaxed hypothesis pave the way to new applications. The possibility of using unbounded functions is primordial in Dabrowski's application of 1.2 to free entropy in [4].

Üstünel's approach is routed in the study of the perturbations of the identity of \mathbb{W} , which is the co-ordinate process, and their invertibility. The question of the invertibility of an adapted perturbation of the identity is linked to the representability of measures and was put to light by the celebrated example of Tsirelson [17]. Üstünel proved that if $u \in L^2(\mu, H)$ and its density is adapted, $I_{\mathbb{W}} + u$ is μ -a.s. invertible if and only if

$$H((I_{\mathbb{W}} + u)\mu|\mu) = \frac{1}{2} \mathbb{E}_\mu [|u|_H^2]$$

To prove 1.2 with the integrability conditions specified above, Üstünel uses the fact that H - C^1 shifts, meaning shifts that are a.s. Fréchet-differentiable on \mathbb{H} with a μ -a.s. continuous on \mathbb{H} Fréchet derivative, are a.s. invertible, and that shifts can be approached with H - C^1 shifts using the Ornstein-Uhlenbeck semigroup.

In [9] we give a variational formulation similar as 1.2 for diffusions solutions of stochastic differential equations, while lowering the integrability hypothesis on f .

In [10] we present a general framework to be able to similarly derive a variational formulation for $-\log \mathbb{E}_\nu [e^{-f}]$ for a large class of measures ν , without increasing the integrability hypothesis on f . We give a set of conditions so that a set of processes (W^u) can act as perturbations of W and allow a Girsanov-like change of variable with respect to a Brownian motion β . We write $\frac{e^{-f}}{\mathbb{E}[e^{-f}]}$ as the

Wick exponential of some v , and then approach v to obtain invertible perturbations of the identity. Hyndman and Wang proved in [11] that

$$(1.3) \quad -\log \mathbb{E}_\mu [e^{-f} | \mathcal{F}_t] = \inf_\theta \left(\mathbb{E}_\theta [f | \mathcal{F}_t] + \mathbb{E}_\theta \left[\log \frac{d\theta}{d\mu} \middle| \mathcal{F}_t \right] \right)$$

where the infimum is taken over the probability measures θ which are absolutely continuous with respect to μ and verify $\mathbb{E}_\mu \left[\frac{d\theta}{d\mu} | \mathcal{F}_t \right] = 1$. They link it to forward-backward stochastic differential equations and apply it to various pricing problems for zero-coupon bonds.

The relation 1.3, obtained for a deterministic time t , is very similar to 1.1 so three questions arise naturally: can we obtain a relation similar to 1.2 for the conditional expectation, can we extend it to other measures with the framework we developed in our third paper, and finally are these relations still valid if we substitute t with a stopping time τ ? Our paper answers affirmatively to these three questions. We keep the notations from our [10] and we prove that

$$(1.4) \quad -\log \mathbb{E}_\nu [e^{-f} | \mathcal{F}_\tau] = \inf_\theta \left(\mathbb{E}_\nu [f | \mathcal{F}_\tau] + \mathbb{E}_\nu \left[\frac{d\theta}{d\nu} \log \frac{d\theta}{d\nu} \middle| \mathcal{F}_\tau \right] \right)$$

$$(1.5) \quad -\log \mathbb{E}_\nu [e^{-f} | \mathcal{F}_\tau] = \inf_u \mathbb{E}_\nu \left[f \circ W^u + \frac{1}{2} |u|_H^2 \middle| \mathcal{F}_\tau \right]$$

In 1.4 we assume that $\mathbb{E}_\nu [fe^{-f}] < \infty$ and the infimum is taken over the probability measures θ on \mathbb{W} which are absolutely continuous with respect to ν and such that $\mathbb{E}_\nu \left[\frac{d\theta}{d\nu} | \mathcal{F}_t \right] = 1$. In 1.5, the infimum is taken over the u from W to H , with adapted density, which are in L^2 and such that $1_{t \leq \tau} \dot{u}(t) = 0$, and we assume that $\mathbb{E}_\nu [fe^{-f}] < \infty$ and that there exists two conjugate integers p and q such that $f \in L^p(\nu)$ and $e^{-f} \in L^q(\nu)$. Observe that we had to increase the integrability hypothesis on f from what we had for the non-conditional case. In fact the integrability hypothesis on f here are the same as in [20]. Finally, we discuss the attainability of the infimum in 1.5 and we obtain an analog of Prékopa-Leindler type theorem for the conditional expectation with respect to μ . However, similarly as in [10], the convexity hypothesis seem quite restrictive.

2. Framework

Set $n \in \mathbb{N}^*$, we denote $\mathbb{W} = C([0, 1], \mathbb{R}^n)$ the canonical Wiener space, $H = \left\{ \int_0^\cdot \dot{h}(s) ds, \dot{h} \in L^2([0, 1]) \right\}$ the associated Cameron-Martin space and W is the coordinate process. We denote (\mathcal{F}_t) its filtration. Set τ a stopping time.

We assume that \mathbb{W} is equipped with a probability measure ν . For $p \geq 0$, we denote

$$L_a^p(\nu, H) = \{u \in L^p(\nu, H), u \text{ is } (\mathcal{F}_t) - \text{adapted}\}$$

and

$$\mathcal{D} = \{u \in L_a^0(\nu, H), u \text{ is } d\nu \times dt - \text{a.s. bounded}\}$$

For $t \in [0, 1]$, we define

$$\pi_t : u \in L_a^0(\nu, H) \mapsto \int_0^\cdot \dot{h}(s) 1_{s \leq t} ds$$

Similarly, we define

$$\begin{aligned}\pi_\tau & : u \in L_a^0(\nu, H) \mapsto \int_0^\cdot \dot{h}(s) 1_{s \leq \tau} ds \\ I - \pi_\tau & : u \in L_a^0(\nu, H) \mapsto \int_0^\cdot \dot{h}(s) 1_{s > \tau} ds\end{aligned}$$

Notice that

$$\begin{aligned}\pi_\tau \mathcal{D} & \subset \mathcal{D} \\ (I - \pi_\tau) \mathcal{D} & \subset \mathcal{D}\end{aligned}$$

and define

$$\mathcal{D}_\tau = (I - \pi_\tau) \mathcal{D}$$

The filtration of a process m will be denoted (\mathcal{F}_t^m) , the filtration of W will be simply denoted (\mathcal{F}_t) . Except if stated otherwise, every filtration considered is completed with respect to ν . If m is a martingale and v has a density whose stochastic integral with respect to m is well defined we will denote

$$\delta_m v = \int_0^1 \dot{v}(s) dm(s)$$

We also denote the Wick exponential as follow

$$\rho(\delta_m v) = \exp \left(\int_0^1 \dot{v}(s) dm(s) - \frac{1}{2} \int_0^1 |\dot{v}(s)|^2 d\langle m \rangle(s) \right)$$

and for $p \geq 0$ we denote

$$G_p(\nu, m) = \{u \in L_a^p(\nu, H), \mathbb{E}_\nu [\rho(-\delta_m u)] = 1\}$$

We assume there exists a family of adapted processes $(W^u)_{u \in \mathcal{D}}$ and a ν -Brownian motion β which verify the following conditions:

- (i) β is a ν -Brownian motion whose canonical filtration is identical to the canonical filtration of W
- (ii) $W^0 = W$
- (iii) For every $u \in \mathcal{D}$, the law of W^u under $\tilde{\nu}^u$ is the same as the law of W under ν , where $\tilde{\nu}^u$ is defined by $\frac{d\tilde{\nu}^u}{d\nu} = \rho(-\delta_\beta u)$
- (iv) For every $u \in \mathcal{D}$,

$$\beta \circ W^u = \beta + u$$

- (v) For every $u, v \in \mathcal{D}$,

$$W^u \circ W^v = W^{v+u \circ W^v} \quad \nu - a.s.$$

- (vi) For every $u \in \mathcal{D}$

$$(W^u(s \wedge \tau), s \leq 1) = (W^{\pi_\tau u}(s \wedge \tau), s \leq 1)$$

Remark: Clearly $\mathcal{D} \subset L_a^\infty(\nu, H)$, so if $u \in \mathcal{D}$, $\mathbb{E}_\nu [\rho(-\delta_\beta u)] = 1$ and $\tilde{\nu}^u$ which was defined in condition (iii) is indeed a probability measure.

Condition (iii) can be written as follow:

Proposition 1. *Set $u \in \mathcal{D}$, for every bounded measurable function f , we have:*

$$\mathbb{E}_\nu [f] = \mathbb{E}_\nu [f \circ W^u \rho(-\delta_\beta u)]$$

Next proposition ensures that the compositions written in (iv) and (v) are well defined.

Proposition 2. *Set $u \in \mathcal{D}$, we have*

$$W^u \nu \sim \nu$$

Proof: Set $f \in C_b(\mathbb{W})$ bounded and measurable, we have, using proposition 1

$$\begin{aligned} \mathbb{E}_{W^u \tilde{\nu}^u} [f] &= \mathbb{E}_{\tilde{\nu}^u} [f \circ W^u] \\ &= \mathbb{E}_{\nu} [f \circ W^u \rho(-\delta_{\beta} u)] \\ &= \mathbb{E}_{\nu} [f] \end{aligned}$$

so $W^u \tilde{\nu}^u = \nu$.

Since $\tilde{\nu} \sim \nu$, we have $W^u \tilde{\nu} \sim W^u \nu$ which conclude the proof. \square

Definition 1. *Set $\tilde{\mathcal{D}}$ a subset of $G_0(\nu, \beta)$ such that the map $u \in \mathcal{D} \mapsto W^u$ can be extended to $\tilde{\mathcal{D}}$ while verifying the following conditions.*

- (i) $\mathcal{D} \subset \tilde{\mathcal{D}} \subset G_2(\nu, \beta)$
- (ii) *For any $u \in \tilde{\mathcal{D}}$, W^u is adapted.*
- (iii) *For every $u \in \tilde{\mathcal{D}}$, the law of W^u under $\tilde{\nu}^u$ is the same as the law of W under ν , where $\tilde{\nu}^u$ is defined by $\frac{d\tilde{\nu}}{d\nu} = \rho(-\delta_{\beta} u)$*
- (iv) *For every $u \in \tilde{\mathcal{D}}$,*

$$\beta \circ W^u = \beta + u$$

- (v) *For every $u, v \in \tilde{\mathcal{D}}$ such that $v + u \circ W^v \in \tilde{\mathcal{D}}$*

$$W^u \circ W^v = W^{v+u \circ W^v} \nu - a.s.$$

- (vi) *There exists $\tilde{\tilde{\mathcal{D}}}$ such that $D'' \subset \tilde{\tilde{\mathcal{D}}} \subset L_a^0(\nu, H)$, $\tilde{\mathcal{D}} = \tilde{\tilde{\mathcal{D}}} \cap G_2(\nu, \beta)$ and for every $u \in \tilde{\mathcal{D}}$ such that the equation $u + v \circ W^u$ has a solution in $G_0(\nu, \beta)$, this equation has a solution in $\tilde{\tilde{\mathcal{D}}}$.*
- (vii) *For every $u \in \tilde{\mathcal{D}}$ such that $\pi_{\tau} u \in \tilde{\mathcal{D}}$,*

$$(W^u(s \wedge \tau), s \leq 1) = (W^{\pi_{\tau} u}(s \wedge \tau), s \leq 1)$$

Remark: \mathcal{D} verify the set of condition above.

Proposition 3. *Set $u \in \tilde{\mathcal{D}}$. For every bounded measurable function f , we have*

$$\mathbb{E}_{\nu} [f] = \mathbb{E}_{\nu} [f \circ W^u \rho(-\delta_{\beta} u)]$$

Furthermore,

$$W^u \nu \sim \nu$$

Proof: The first assertion is condition (iii). The proof of the second assertion is the same as the case $u \in \mathcal{D}$. \square

Definition 2. We define $\tilde{\mathcal{D}}_\tau$ as

$$\tilde{\mathcal{D}}_\tau = \tilde{\mathcal{D}} \cap (I - \pi_\tau)L_a^0(\nu, H)$$

3. Conditional expectation results

We need the abstract Bayes formula for a stopping time:

Lemma 1. Set θ a probability measure on $(\mathbb{W}, \mathcal{F})$ such that $\theta \ll \nu$. Denote

$$L = \frac{d\theta}{d\nu}$$

For every measurable $f : \mathbb{W} \rightarrow \mathbb{R}$ we have

$$\mathbb{E}_\theta[f|\mathcal{F}_\tau] = \frac{\mathbb{E}_\nu[fL|\mathcal{F}_\tau]}{\mathbb{E}_\nu[L|\mathcal{F}_\tau]}$$

Proof: We can assume f is positive. Denote for $s \in [0, 1]$

$$L(s) = \mathbb{E}_\nu[L|\mathcal{F}_s]$$

The martingale stopping theorem gives

$$L(\tau) = \mathbb{E}_\nu[L|\mathcal{F}_\tau]$$

Set $A \in \mathcal{F}_\tau$, we need to prove that

$$\mathbb{E}_\nu[1_A L(\tau) \mathbb{E}_\theta[f|\mathcal{F}_\tau]] = \mathbb{E}_\nu[1_A \mathbb{E}_\nu[fL|\mathcal{F}_\tau]]$$

We have

$$\begin{aligned} \mathbb{E}_\nu[1_A L(\tau) \mathbb{E}_\theta[f|\mathcal{F}_\tau]] &= \mathbb{E}_{\nu|_{\mathcal{F}_\tau}}[1_A L(\tau) \mathbb{E}_\theta[f|\mathcal{F}_\tau]] \\ &= \mathbb{E}_{\theta|_{\mathcal{F}_\tau}}[1_A \mathbb{E}_\theta[f|\mathcal{F}_\tau]] \\ &= \mathbb{E}_\theta[1_A \mathbb{E}_\theta[f|\mathcal{F}_\tau]] \\ &= \mathbb{E}_\theta[1_A f] \\ &= \mathbb{E}_\nu[1_A f L] \\ &= \mathbb{E}_\nu[1_A \mathbb{E}_\nu[fL|\mathcal{F}_\tau]] \end{aligned}$$

□

Proposition 4. Set $u \in \tilde{\mathcal{D}}$ and $f \in L^0(\nu)$ an \mathcal{F}_τ -measurable function. Then ν -a.s.

$$f \circ W^u = f \circ W^{\pi_\tau u}$$

Consequently, if $u \in \tilde{\mathcal{D}}_\tau$, we have ν -a.s.

$$f \circ W^u = f$$

Proof: For $s \in [0, 1]$, we have

$$\begin{aligned} \beta(s) \circ W^u &= \beta(s) + u(s) \\ &= \beta(s) + \pi_s u(s) \\ &= \beta(s) \circ W^{\pi_s u} \end{aligned}$$

Consequently, for $h \in H$

$$\rho(\delta_\beta \pi_s h) \circ W^u = \rho(\delta_\beta \pi_s h) \circ W^{\pi_s u}$$

and

$$\rho(\delta_\beta \pi_\tau h) \circ W^u = \rho(\delta_\beta \pi_\tau h) \circ W^{\pi_\tau u}$$

Denote

$$L^2(\nu, \mathcal{F}_\tau) = \{f \in L^2(\nu), f \text{ is } \mathcal{F}_\tau\text{-measurable}\}$$

$(\rho(\delta_\beta \pi_s h), s \in [0, 1])$ being a closed martingale, we have

$$\mathbb{E}_\nu [\rho(\delta_\beta h) | \mathcal{F}_\tau] = \rho(\delta_\beta \pi_\tau h)$$

Since β and W have the same filtration, the vector space generated by $\{\rho(\delta_\beta h), h \in H\}$ is dense in $L^2(\nu)$. $g \in L^2(\nu) \mapsto \mathbb{E}_\nu [g | \mathcal{F}_\tau]$ being a continuous surjection from $L^2(\nu)$ to $L^2(\nu, \mathcal{F}_\tau)$, the vector space generated by $\{\rho(\delta_\beta \pi_\tau h), h \in H\}$ is dense in $L^2(\nu, \mathcal{F}_\tau)$. We denote E this vector space.

Assume that f is bounded, there exists $(f_n) \in E^\mathbb{N}$ which converges to f ν -a.s. Since $W^u \nu \ll \nu$ and $W^{\pi_\tau u} \nu \ll \nu$, $(f_n \circ W^u)$ converges ν -a.s. to $f \circ W^u$ and $(f_n \circ W^{\pi_\tau u})$ converges ν -a.s. to $f \circ W^{\pi_\tau u}$, which ensures the result in this case

Finally, if f is only supposed to be \mathcal{F}_τ -measurable, there exists a sequence of bounded \mathcal{F}_τ -measurable functions which converges to f and we proceed as above. \square

Proposition 5. *Set L a density on $(\mathbb{W}, \nu, \mathcal{F})$ such that $L > 0$ ν -a.s. Denote,*

$$M(s) = \mathbb{E}_\nu [L | \mathcal{F}_s]$$

and set $v \in L_a^0(\nu, H)$ such that

$$M(s) = \rho(-\delta_\beta \pi_s v)$$

Then the two following propositions are equivalent:

- (i) $\pi_\tau v = 0$ ν -a.s.
- (ii) $\mathbb{E}_\nu [L | \mathcal{F}_\tau] = 1$ ν -a.s.

Proof: The direct implication is trivial. Conversely, M is a martingale with unit expectation and since

$$M(s) = 1 + \int_0^s M(r) \dot{v}(r) d\beta(r)$$

We have

$$\langle M - 1 \rangle = \int_0^\cdot (M(r) \dot{v}(r))^2 dr$$

Proposition(ii) gives $(M(s \wedge \tau) - 1, s \leq 1) = 0$ ν -a.s., so $(\langle M - 1 \rangle(s \wedge \tau), s \leq 1) = 0$ ν -a.s., $L > 0$ so ν -a.s. $M(s) > 0$ for any $s \in [0, 1]$ and we have proposition (i). \square

Lemma 2. *Set $u \in \tilde{\mathcal{D}}_\tau$ and denote $L = \frac{dW^u \nu}{d\nu}$. We have*

$$\mathbb{E}_\nu [L | \mathcal{F}_\tau] = 1$$

Consequently, for any $f \in L^1(W^u \nu)$, we have

$$\mathbb{E}_{W^u \nu} [f | \mathcal{F}_\tau] = \mathbb{E}_\nu [f \circ W^u | \mathcal{F}_\tau] = \mathbb{E}_\nu [f L | \mathcal{F}_\tau]$$

Proof: Set $B \in \mathcal{F}_\tau$, we have

$$\mathbb{E}_\nu [1_B L] = \mathbb{E}_\nu [1_B \circ W^u] = \mathbb{E}_\nu [1_B \circ W^{\pi_\tau u}] = \mathbb{E}_\nu [1_B]$$

Then the second assertion is a direct consequence of Bayes formula. \square

4. Invertibility results

Definition 3. A measurable map $U : \mathbb{W} \rightarrow \mathbb{W}$ is said to be ν -a.s. left-invertible if and only if $U\nu \ll \nu$ and there exists a measurable map $V : \mathbb{W} \rightarrow \mathbb{W}$ such that $V \circ U = I_{\mathbb{W}}$ ν -a.s.

A measurable map $U : \mathbb{W} \rightarrow \mathbb{W}$ is said to be ν -a.s. right-invertible if and only if there exists a measurable map $V : \mathbb{W} \rightarrow \mathbb{W}$ such that $V\nu \ll \nu$ and $U \circ V = I_{\mathbb{W}}$ ν -a.s.

Proposition 6. Set $U, V : \mathbb{W} \rightarrow \mathbb{W}$ measurable maps such that $V \circ U = I_{\mathbb{W}}$ ν -a.s. and $V\nu \ll \nu$. Then $U \circ V = I_{\mathbb{W}}$ $U\nu$ -a.s., so if $U\nu \sim \nu$, we also have $U \circ V = I_{\mathbb{W}}$ ν -a.s. In that case, we will say that U is ν -a.s. invertible and we also have $V\nu \sim \nu$.

Proof: See [10]. \square

Proposition 7. Set $u \in \tilde{\mathcal{D}}_\tau \cap L_a^2(\nu, H)$. If W^u is ν -a.s. left-invertible. Then there exists $v \in \tilde{\mathcal{D}}_\tau$ such that ν -a.s.

$$W^v \circ W^u = W^u \circ W^v = I_{\mathbb{W}}$$

and

$$\begin{aligned} \frac{dW^u \nu}{d\nu} &= \rho(-\delta_\beta v) \\ \frac{dW^v \nu}{d\nu} &= \rho(-\delta_\beta u) \end{aligned}$$

Proof: Everything is already known from [10] except the fact that $\pi_\tau v = 0$. This arises from the relation

$$\dot{v}(s) = -\dot{u}(s) \circ W^v$$

\square

Now we recall two very useful lemmas, see [10] for the proof

Lemma 3. Set $u \in \tilde{\mathcal{D}} \cap L_a^2(\nu, H)$ and denote $L = \frac{dW^u \nu}{d\nu}$, we have ν -a.s.

$$L \circ W^u \mathbb{E}_\nu \left[\rho(-\delta_\beta u) \middle| \mathcal{F}_1^{W^u} \right] = 1$$

Theorem 1. Set $u \in \tilde{\mathcal{D}} \cap L_a^2(\nu, H)$ and denote $L = \frac{dW^u \nu}{d\nu}$. Then W^u is ν -a.s. left-invertible if and only if

$$\mathbb{E}_\nu [L \log L] = \frac{1}{2} \mathbb{E}_\nu [|u|_H^2]$$

Moreover, if W^u is ν -a.s. left-invertible, we have ν -a.s.

$$L \circ W^u \rho(-\delta_\beta u) = 1$$

Now we give the results relative to the invertibility of W^u when $\pi_\tau u = 0$.

Proposition 8. Set $u \in \tilde{\mathcal{D}}_\tau \cap L_a^2(\nu, H)$ and denote $L = \frac{dW^u \nu}{d\nu}$. We have ν -a.s.:

$$\mathbb{E}_\nu [L \log L | \mathcal{F}_\tau] \leq \frac{1}{2} \mathbb{E}_\nu [|u|_H^2 | \mathcal{F}_\tau]$$

Proof: We have $(W^u(s \wedge \tau))_{s \leq 1} = (W(s \wedge \tau))_{s \leq 1}$ hence

$$\mathcal{F}_\tau = \mathcal{F}_\tau^{W^u} \subset \mathcal{F}_1^{W^u}$$

Consequently, using lemma 3 and Jensen inequality, we have:

$$\begin{aligned} \mathbb{E}_\nu [L \log L | \mathcal{F}_\tau] &= \mathbb{E}_\nu [\log L \circ W^u | \mathcal{F}_\tau] \\ &\leq -\mathbb{E}_\nu \left[\log \mathbb{E}_\nu \left[\rho(-\delta_\beta u) \middle| \mathcal{F}_1^{W^u} \right] \middle| \mathcal{F}_\tau \right] \\ &\leq -\mathbb{E}_\nu \left[\mathbb{E}_\nu \left[\log \rho(-\delta_\beta u) \middle| \mathcal{F}_1^{W^u} \right] \middle| \mathcal{F}_\tau \right] \\ &\leq -\mathbb{E}_\nu [\log \rho(-\delta_\beta u) | \mathcal{F}_\tau] \\ &\leq \frac{1}{2} \mathbb{E}_\nu [|u|_H^2 | \mathcal{F}_\tau] \end{aligned}$$

□

Theorem 2. Set $u \in \tilde{\mathcal{D}}_\tau \cap L_a^2(\nu, H)$ and denote $L = \frac{dW^u \nu}{d\nu}$. Then W^u is ν -a.s. left-invertible, if and only if ν -a.s.

$$\mathbb{E}_\nu [L \log L | \mathcal{F}_\tau] = \frac{1}{2} \mathbb{E}_\nu [|u|_H^2 | \mathcal{F}_\tau]$$

Proof: Assume that the equality holds, taking the expectation we have

$$\mathbb{E}_\nu [L \log L] = \frac{1}{2} \mathbb{E}_\nu [|u|_H^2]$$

so according to theorem 1 W^u is ν -a.s. left invertible.

Conversely, using again theorem 1, we have

$$L \circ W^u \rho(-\delta_\beta u) = 1$$

So, since $\pi_\tau u = 0$,

$$\begin{aligned} \mathbb{E}_\nu [L \log L | \mathcal{F}_\tau] &= \mathbb{E}_\nu [\log L \circ W^u | \mathcal{F}_\tau] \\ &= \mathbb{E}_\nu [-\log \rho(-\delta_\beta u) | \mathcal{F}_\tau] \\ &= \frac{1}{2} \mathbb{E}_\nu [|u|_H^2 | \mathcal{F}_\tau] \end{aligned}$$

□

Definition 4. We denote

$$\begin{aligned} \mathcal{D}^i &= \{u \in \mathcal{D}, W^u \text{ is } \nu - \text{a.s. invertible}\} \\ \mathcal{D}_\tau^i &= \mathcal{D}_\tau \cap \mathcal{D}^i \end{aligned}$$

5. Approximation of absolutely continuous measures

Theorem 3. *If $\theta \sim \nu$ is such that there exists $p > 1$ such that*

$$\frac{d\theta}{d\nu} \in L^p(\nu)$$

and ν -a.s.

$$\mathbb{E}_\nu \left[\frac{d\theta}{d\nu} \middle| \mathcal{F}_\tau \right] = 1$$

There exists $(u_n) \in (\mathcal{D}_\tau^i)^\mathbb{N}$ such that,

$$\frac{dW^{u_n}\nu}{d\nu} \rightarrow \frac{d\theta}{d\nu} \text{ in } L^p(\nu)$$

Proof: Eventually sequentializing afterward, we have to prove that for any $\epsilon > 0$, there exists $u \in \mathcal{D}_\tau^i$ such that

$$\left| \frac{dW^u\nu}{d\nu} - \frac{d\theta}{d\nu} \right|_{L^p(\nu)} \leq \epsilon$$

The proof is divided in six steps.

Step 1 : We approximate $\frac{d\theta}{d\nu}$ with a density that is both lower and upper bounded.

Denote

$$L(s) = \mathbb{E}_\nu \left[\frac{d\theta}{d\nu} \middle| \mathcal{F}_s \right]$$

and for $n \in \mathbb{N}$,

$$T_n = \inf \{s \in [0, 1], L(s) \geq n\}$$

$L(1) = \frac{d\theta}{d\nu}$ and L being a closed martingale, $L^{T_n}(\cdot)$ is still a closed martingale which converges in L^1 to $L(T_n) = \mathbb{E}_\nu [L(1) | \mathcal{F}_{T_n}]$, so

$$\begin{aligned} \mathbb{E}_\nu [L(T_n) | \mathcal{F}_\tau] &= \mathbb{E}_\nu [L^{T_n}(1) | \mathcal{F}_\tau] \\ &= L^{T_n}(\tau) \\ &= L(\tau \wedge T_n) \\ &= \mathbb{E}_\nu [L(1) | \mathcal{F}_{\tau \wedge T_n}] \\ &= \mathbb{E}_\nu [\mathbb{E}_\nu [L(1) | \mathcal{F}_\tau] | \mathcal{F}_{\tau \wedge T_n}] \\ &= 1 \end{aligned}$$

Furthermore, since L is a closed martingale, $(\mathbb{E}_\nu [L(1) | \mathcal{F}_{T_n}])_{n \in \mathbb{N}}$ is also a closed martingale so is uniformly integrable. $(L(T_n))$ converges to L_1 ν -a.s. and Jensen inequality gives

$$0 \leq L(T_n)^p = \mathbb{E}_\nu [L(1)^p | \mathcal{F}_{T_n}] \leq \mathbb{E}_\nu [L(1)^p]$$

So $(L(T_n)^p)$ is uniformly integrable and $(L(T_n))$ converges in $L^p(\nu)$ to $L(1)$, so there exists $n_0 \in \mathbb{N}$ such that

$$|L(T_{n_0}) - L(1)|_{L^p(\nu)} \leq \epsilon$$

$\left(\frac{L(T_{n_0}) + a}{1+a} \right)$ converges ν -a.s. to $L(T_{n_0})$ when a converges to 0. Set $a \in [0, 1]$, we have

$$0 \leq \frac{L(T_{n_0}) + a}{1+a} \leq L(T_{n_0}) + 1$$

$L(T_{n_0}) + 1 \in L^p(\nu)$ so according to the Lebesgue theorem, $\left(\frac{L(T_{n_0})+a}{1+a}\right)$ converges to $L(T_{n_0})$ in $L^p(\nu)$ and there exists $a \in [0, 1]$ such that

$$\left| \frac{L(T_{n_0}) + a}{1 + a} - L(T_{n_0}) \right|_{L^p(\nu)} \leq \epsilon$$

$\frac{L(T_{n_0})+a}{1+a}$ is both lower-bounded and upper-bounded in $L^\infty(\nu)$, denote these bounds respectively d and D .

Denote

$$M(s) = \mathbb{E}_\nu \left[\frac{L(T_{n_0}) + a}{1 + a} \middle| \mathcal{F}_s \right]$$

We can write

$$M = \exp \left(\int_0^\cdot \dot{\alpha}(r) d\beta(r) - \frac{1}{2} \int_0^\cdot |\dot{\alpha}(r)|^2 dr \right)$$

with $\alpha \in (I - \pi_\tau)L_a^0(\nu, H)$ since

$$\mathbb{E}_\nu [M(1) | \mathcal{F}_\tau] = \frac{\mathbb{E}_\nu [L_{T_{n_0}} | \mathcal{F}_\tau] + a}{1 + a} = 1$$

Step 2 : we prove that $\alpha \in (I - \pi_\tau)L_a^2(\nu, H)$

Set

$$S_n = \inf \left\{ s \in [0, 1], \int_0^s |\dot{\alpha}(r)|^2 dr > n \right\}$$

(S_n) is a sequence of stopping times which increases stationarily toward 1. We have, using $M = 1 + \int_0^\cdot \dot{\alpha}(r) M(r) d\beta(r)$

$$\begin{aligned} \mathbb{E}_\nu \left[(M(s \wedge S_n) - 1)^2 \right] &= \mathbb{E}_\nu \left[\int_0^{s \wedge S_n} |\dot{\alpha}(r)|^2 M(r)^2 dr \right] \\ &\geq d^2 \mathbb{E}_\nu \left[\int_0^{s \wedge S_n} |\dot{\alpha}(r)|^2 dr \right] \end{aligned}$$

so

$$\mathbb{E}_\nu \left[\int_0^{s \wedge S_n} |\dot{\alpha}(r)|^2 dr \right] \leq \frac{1}{d^2} \mathbb{E}_\nu \left[(M(s \wedge S_n) - 1)^2 \right] \leq \frac{2(D^2 + 1)}{d^2}$$

hence passing to the limit

$$\mathbb{E}_\nu \left[\int_0^1 |\dot{\alpha}(r)|^2 dr \right] \leq \infty$$

Step 3 : We approximate α with an element of $(I - \pi_\tau)L_a^\infty(\nu, H)$.

Define

$$\alpha^n(s, w) \in [0, 1] \times \mathbb{W} \mapsto \int_0^s \dot{\alpha}(r, w) 1_{[0, S_n]}(r, w) dr$$

and

$$M^n(s) = \exp \left(\int_0^s \dot{\alpha}^n(r) d\beta(r) - \frac{1}{2} \int_0^s |\dot{\alpha}^n(r)|^2 dr \right)$$

$\alpha^n \in (I - \pi_\tau)L^\infty(\nu, H)$ and $M^n(1) = \mathbb{E}_\nu [M(1) | \mathcal{F}_{S_n}]$ so $(M^n(1))_{n \in \mathbb{N}}$ is a closed martingale since M is one, hence it converges ν -a.s. to $M(1)$ and it is uniformly integrable. Jensen inequality gives

$$0 \leq |M^n(1)|^p \leq \mathbb{E}_\nu [M(1) | \mathcal{F}_{S_n}]^p \leq \mathbb{E}_\nu [M(1)^p | \mathcal{F}_{S_n}]$$

So $(|M^n(1)|^r)$ is uniformly integrable and $(M^n(1))$ converges to $M(1)$ in $L^p(\nu)$. Consequently, there exists $n \in \mathbb{N}$ such that

$$|M^n(1) - M(1)|_{L^p(\nu)} \leq \epsilon$$

Step 4 : we approximate α^n with an element of \mathcal{D}_τ

Define

$$\xi^{n,m} : (s, w) \in [0, 1] \times \mathbb{W} \mapsto \int_0^s \max(\min(\dot{\alpha}^n(r, w), m), -m) dr$$

and

$$M^{n,m}(s) = \exp \left(\int_0^s \xi^{n,m}(r) d\beta(s) - \frac{1}{2} \int_0^s |\xi^{n,m}(r)|^2 dr \right)$$

$\xi^{n,m} \in \mathcal{D}_\tau$ and $(M^{n,m}(1))$ and converges to $M^n(1)$ in probability. To prove that $((M^{n,m}(1))^p)$ is uniformly integrable, it is sufficient to prove that $(M^{n,m}(1))$ is bounded in every $L^q(\nu)$ with $q > 1$. Set $q > 1$

$$\begin{aligned} \mathbb{E}_\nu [|M_1^{n,m}|^q] &= \mathbb{E}_\nu \left[\exp \left(q \int_0^1 \xi^{n,m}(s) d\beta(s) - \frac{q}{2} \int_0^1 |\xi^{n,m}(s)|^2 ds \right) \right] \\ &= \mathbb{E}_\nu \left[\exp \left(q \int_0^1 \xi^{n,m}(s) d\beta(s) - \frac{q^2}{2} \int_0^1 |\xi^{n,m}(s)|^2 ds \right) \exp \left(\frac{q^2 - q}{2} \int_0^1 |\xi^{n,m}(s)|^2 ds \right) \right] \\ &\leq \mathbb{E}_\nu \left[\exp \left(\int_0^1 q \xi^{n,m}(s) d\beta(s) - \frac{1}{2} \int_0^1 |p \xi^{n,m}(s)|^2 ds \right) \exp \left(\frac{q^2 - q}{2} n \right) \right] \\ &\leq \exp \left(\frac{q^2 - q}{2} n \right) \end{aligned}$$

so $(M^{n,m}(1), m \in \mathbb{N})$ converges to M_1^n in $L^p(\nu)$ and there exists some $m > 0$ such that

$$|M^{n,m}(1) - M^n(1)|_{L^p(\nu)} \leq \epsilon$$

Step 5 : We approximate $\xi^{n,m}$ with a retarded shift γ^η , so that W^{γ^η} is ν -a.s. invertible.

For a given $\eta > 0$, set

$$\gamma^\eta(s, w) \in [0, 1] \times W \mapsto \int_0^s \xi^{n,m}(r - \eta) 1_{r \geq \eta} ds$$

and

$$N^\eta(s) = \exp \left(\int_0^s \gamma^\eta(r) d\beta(r) - \frac{1}{2} \int_0^s |\gamma^\eta(r)|^2 dr \right)$$

Clearly $\gamma^\eta \in \mathcal{D}_\tau$ and $\gamma^\eta \rightarrow \xi^{n,m}$ in $L^2(\nu, H)$ when $\eta \rightarrow 0$, which ensures that $(N^\eta(1), \eta > 0)$ converges to $M^{n,m}(1)$ in probability.

As in step 4, $(N^\eta(1), \eta > 0)$ is bounded in every $L^q(\nu)$ and so $(N^\eta(1)^p, \eta > 0)$ is uniformly integrable and $(N^\eta(1), \eta > 0)$ converges to $M^{n,m}(1)$ in $L^p(\nu)$. There exists $\eta > 0$ such that

$$|N^\eta(1) - M^{n,m}(1)|_{L^p(\nu)} \leq \epsilon$$

Using triangular inequality, we have

$$\begin{aligned}
 \left| \frac{d\theta}{d\nu} - N^\eta(1) \right|_{L^p(\nu)} &\leq \left| \frac{d\theta}{d\nu} - L(T_{n_0}) \right|_{L^p(\nu)} + \left| L(T_{n_0}) - \frac{L(T_{n_0}) + a}{1+a} \right|_{L^p(\nu)} \\
 &\quad + \left| \frac{L(T_{n_0}) + a}{1+a} - M^n(1) \right|_{L^p(\nu)} \\
 &\quad + |M^n(1) - M^{n,m}(1)|_{L^p(\nu)} \\
 &\quad + |M^{n,m}(1) - N^\eta(1)|_{L^p(\nu)} \\
 &\leq 5\epsilon
 \end{aligned}$$

Step 6 : We prove that $W^{-\gamma^\eta}$ is ν -a.s. left-invertible and is the solution to our problem.

Set $A \subset \mathbb{W}$ such that $\nu(A) = 1$ and for every $w \in A$, $\beta \circ W^{-\gamma^\eta}(w) = \beta(w) - \gamma^\eta(w)$ and set $w_1, w_2 \in A$ such that $W^{-\gamma^\eta}(w_1) = W^{-\gamma^\eta}(w_2)$. We have

$$\begin{aligned}
 \beta \circ W^{-\gamma^\eta}(w_1) &= \beta \circ W^{-\gamma^\eta}(w_2) \\
 \beta(w_1) - \int_0^\cdot \dot{\gamma}^\eta(s, w_1) ds &= \beta(w_2) - \int_0^\cdot \dot{\gamma}^\eta(s, w_2) ds
 \end{aligned}$$

For any $s \in [0, \eta]$, $\beta(s, w_1) = \beta(s, w_2)$, γ^η being adapted to filtration $(\mathcal{F}_{s-\eta}^\beta)$, it implies that for $s \in [0, 2\eta]$

$$\int_0^s \dot{\gamma}^\eta(r, w_1) ds = \int_0^s \dot{\gamma}^\eta(r, w_2) ds$$

and

$$\beta(s, w_1) = \beta(s, w_2)$$

An easy iteration shows that $\beta(w_1) = \beta(w_2)$.

Since β and W have the same filtrations and β is μ -a.s. path-continuous, we can write $W(t) = \phi_t(\beta(s), s \in [0, t] \cap \mathbb{Q})$ ν -a.s. for every $t \in [0, 1]$, with ϕ_t a measurable function from $\mathbb{R}^\mathbb{Q}$ to \mathbb{R} , see [13]. Consequently, we can write $(W(t), t \in [0, 1] \cap \mathbb{Q}) = \phi(\beta(t), t \in [0, 1] \cap \mathbb{Q})$ ν -a.s., with ϕ a measurable function from $\mathbb{R}^\mathbb{Q}$ to $\mathbb{R}^\mathbb{Q}$. Denote

$$A' = A \cap \{w \in \mathbb{W}, (W(t, w), t \in [0, 1] \cap \mathbb{Q}) = \phi(\beta(t, w), t \in [0, 1] \cap \mathbb{Q})\}$$

$\nu(A') = 1$. Set $w_1, w_2 \in A'$ such that $W^{-\gamma^\eta}(w_1) = W^{-\gamma^\eta}(w_2)$. We have $\beta(w_1) = \beta(w_2)$ so

$$\begin{aligned}
 (W(t, w_1), t \in [0, 1] \cap \mathbb{Q}) &= (W(t, w_2), t \in [0, 1] \cap \mathbb{Q}) \\
 (w_1(t), t \in [0, 1] \cap \mathbb{Q}) &= (w_2(t), t \in [0, 1] \cap \mathbb{Q})
 \end{aligned}$$

w_1 and w_2 are continuous and coincide on $[0, 1] \cap \mathbb{Q}$ so they are equal.

$W^{-\gamma^\eta}$ is ν -a.s. injective and so ν -a.s. left-invertible, its inverse is of the form W^{v^η} , with $v^\eta \in \mathcal{D}_\tau$ and we have

$$\frac{dW^{v^\eta}\nu}{d\nu} = L_1^{\eta, n}$$

So $W^{v^\eta}\nu \sim \nu$ and

$$W^{v^\eta} \circ W^{-\gamma^\eta} = W^{-\gamma^\eta} \circ W^{v^\eta} \quad \nu - a.s.$$

□

Corollary 1. *If $\theta \sim \nu$ is such that there exists $q, p > 1$ such that $p^{-1} + q^{-1} = 1$ and*

$$\begin{aligned} \frac{d\theta}{d\nu} &\in L^p(\nu) \\ \log \frac{d\theta}{d\nu} &\in L^q(\nu) \end{aligned}$$

and ν -a.s.

$$\mathbb{E}_\nu \left[\frac{d\theta}{d\nu} \middle| \mathcal{F}_\tau \right] = 1$$

there exists $(u_n) \in (\mathcal{D}_\tau^i)^\mathbb{N}$ such that

$$\begin{aligned} \mathbb{E}_\nu \left[\frac{dW^{u_n} \nu}{d\nu} \log \frac{dW^{u_n} \nu}{d\nu} \middle| \mathcal{F}_\tau \right] &\rightarrow \mathbb{E}_\nu \left[\frac{d\theta}{d\nu} \log \frac{d\theta}{d\nu} \middle| \mathcal{F}_\tau \right] \quad \nu - a.s. \\ \mathbb{E}_\nu \left[\frac{dW^{u_n} \nu}{d\nu} \log \frac{d\theta}{d\nu} \middle| \mathcal{F}_\tau \right] &\rightarrow \mathbb{E}_\nu \left[\frac{d\theta}{d\nu} \log \frac{d\theta}{d\nu} \middle| \mathcal{F}_\tau \right] \quad \nu - a.s. \end{aligned}$$

Proof: From theorem 3, there exists $(u_n) \in (\mathcal{D}_\tau^i)^\mathbb{N}$ such that for every n ,

$$\frac{dW^{u_n} \nu}{d\nu} \rightarrow \frac{d\theta}{d\nu} \text{ in } L^r(\nu)$$

This implies

$$\frac{dW^{u_n} \nu}{d\nu} \log \frac{dW^{u_n} \nu}{d\nu} \rightarrow \frac{d\theta}{d\nu} \log \frac{d\theta}{d\nu} \text{ in } L^1(\nu)$$

Holder inequality gives

$$\begin{aligned} \left| \frac{dW^{u_n} \nu}{d\nu} \log \frac{d\theta}{d\nu} - \frac{d\theta}{d\nu} \log \frac{d\theta}{d\nu} \right|_{L^1(\nu)} &\leq \left| \frac{dW^{u_n} \nu}{d\nu} - \frac{d\theta}{d\nu} \right|_{L^p(\nu)} \left| \log \frac{d\theta}{d\nu} \right|_{L^q(\nu)} \\ &\rightarrow 0 \end{aligned}$$

The corresponding conditional expectations converges similarly in $L^1(\nu)$ since $\mathbb{E}_\nu[\cdot | \mathcal{F}_\tau]$ is a bounded operator with norm 1 in $L^1(\nu)$. Finally we can extract a subsequence of (u_n) to get the two desired almost sure convergences. \square

6. Variational problem

As stated in the beginning, we aim to provide a variational representation of $-\log \mathbb{E}_\nu [e^{-f} | \mathcal{F}_\tau]$.

Definition 5. *We denote \mathcal{P}_τ the set of probability measures θ on $(\mathbb{W}, \mathcal{F})$ such that*

$$\begin{aligned} \theta &\sim \nu \\ \mathbb{E}_\nu \left[\frac{d\theta}{d\nu} \middle| \mathcal{F}_\tau \right] &= 1 \end{aligned}$$

Theorem 4. *Set $f : \mathbb{W} \rightarrow \mathbb{R}$ a measurable function verifying*

$$\mathbb{E}_\nu [|f|(1 + e^{-f})] < \infty$$

Then

$$-\log \mathbb{E}_\nu [e^{-f} | \mathcal{F}_\tau] = \inf_{\theta \in \mathcal{P}_\tau} \mathbb{E}_\theta \left[f + \log \frac{d\theta}{d\nu} \middle| \mathcal{F}_\tau \right] \quad \nu - a.s.$$

and the unique infimum is attained at the measure

$$d\theta_0 = \frac{e^{-f}}{\mathbb{E}_\nu [e^{-f} | \mathcal{F}_\tau]} d\nu$$

Proof: Set $\theta \in \mathcal{P}_\tau$, denote

$$L(s) = \frac{d\theta}{d\nu} \Big|_{\mathcal{F}_s}$$

$L(\tau) = 1$ ν -a.s. since $\theta \in \mathcal{P}_\tau$ so using the Bayes formula:

$$\begin{aligned} \log \mathbb{E}_\nu [e^{-f} | \mathcal{F}_\tau] &= \log \mathbb{E}_\nu \left[e^{-f} \frac{L(1)}{L(1)} \Big| \mathcal{F}_\tau \right] \\ &= \log \mathbb{E}_\theta \left[e^{-f} \frac{L(\tau)}{L(1)} \Big| \mathcal{F}_\tau \right] \\ &= \log \mathbb{E}_\theta \left[\frac{e^{-f}}{L(1)} \Big| \mathcal{F}_\tau \right] \end{aligned}$$

Jensen inequality gives

$$\begin{aligned} -\log \mathbb{E}_\theta \left[\frac{e^{-f}}{L(1)} \Big| \mathcal{F}_\tau \right] &\leq \mathbb{E}_\theta \left[-\log \frac{e^{-f}}{L(1)} \Big| \mathcal{F}_\tau \right] \\ &\leq \mathbb{E}_\theta [f | \mathcal{F}_\tau] + \mathbb{E}_\theta [\log L(1) | \mathcal{F}_\tau] \\ &\leq \mathbb{E}_\theta [f | \mathcal{F}_\tau] + \mathbb{E}_\theta [\log L(1) | \mathcal{F}_\tau] \end{aligned}$$

A straightforward calculation gives

$$\mathbb{E}_{\theta_0} \left[f + \log \frac{d\theta_0}{d\nu} \Big| \mathcal{F}_\tau \right] = -\log \mathbb{E}_\nu [e^{-f} | \mathcal{F}_\tau]$$

and the reverse inequality. □

Proposition 9. Set $f : \mathbb{W} \rightarrow \mathbb{R}$ a measurable function verifying $\mathbb{E}_\nu [|f|(1 + e^{-f})] < \infty$, then

$$-\log \mathbb{E}_\nu [e^{-f} | \mathcal{F}_\tau] \leq \inf_{u \in \tilde{\mathcal{D}}_\tau \cap L_a^2(\nu, H)} \mathbb{E}_\nu \left[f \circ W^u + \frac{1}{2} |u|_H^2 \Big| \mathcal{F}_\tau \right] \quad \nu - a.s.$$

Proof: Denote \mathcal{P}'_τ the set of the elements S of \mathcal{P}_τ such that there exists some $u \in \tilde{\mathcal{D}}_\tau$ which verifies $S = W^u \nu$.

Set $\theta \in \mathcal{P}'_\tau$ and denote $L = \frac{d\theta}{d\nu}$. Since $\mathbb{E}_\nu [L | \mathcal{F}_\tau] = 1$, we have using Bayes formula

$$\begin{aligned} \mathbb{E}_\theta [f | \mathcal{F}_\tau] &= \mathbb{E}_\nu [fL | \mathcal{F}_\tau] \\ &= \mathbb{E}_\nu [f \circ W^u | \mathcal{F}_\tau] \\ \mathbb{E}_\theta [\log L | \mathcal{F}_\tau] &= \mathbb{E}_\nu [L \log L | \mathcal{F}_\tau] \\ &\leq \frac{1}{2} \mathbb{E}_\nu [|u|_H^2 | \mathcal{F}_\tau] \end{aligned}$$

So since $\mathcal{P}'_\tau \subset \mathcal{P}_\tau$, we have

$$\begin{aligned} -\log \mathbb{E}_\nu [e^{-f} | \mathcal{F}_\tau] &= \inf_{\theta \in \mathcal{P}_\tau} \left(\mathbb{E}_\theta [f | \mathcal{F}_\tau] + \mathbb{E}_\theta \left[\log \frac{d\theta}{d\nu} \Big| \mathcal{F}_\tau \right] \right) \\ &\leq \inf_{\theta \in \mathcal{P}'_\tau} \left(\mathbb{E}_\theta [f | \mathcal{F}_\tau] + \mathbb{E}_\theta \left[\log \frac{d\theta}{d\nu} \Big| \mathcal{F}_\tau \right] \right) \\ &\leq \inf_{u \in \tilde{\mathcal{D}}_\tau \cap L_a^2(\nu, H)} \mathbb{E}_\nu \left[f \circ W^u + \frac{1}{2} |u|_H^2 \Big| \mathcal{F}_\tau \right] \end{aligned}$$

□

Here is the main result.

Theorem 5. *Set $f : \mathbb{W} \rightarrow \mathbb{R}$ measurable and $p, q > 1$ such that $p^{-1} + q^{-1} = 1$ and $f \in L^p(\nu)$, $e^{-f} \in L^q(\nu)$, then we have*

$$-\log \mathbb{E}_\nu [e^{-f} | \mathcal{F}_\tau] = \inf_{u \in \mathcal{D}_\tau^i} \mathbb{E}_\nu \left[f \circ W^u + \frac{1}{2} |u|_H^2 \middle| \mathcal{F}_\tau \right] \quad \nu - a.s.$$

Proof: The inequality

$$-\log \mathbb{E}_\nu [e^{-f} | \mathcal{F}_\tau] \leq \inf_{u \in \mathcal{D}_\tau^i} \mathbb{E}_\nu \left[f \circ W^u + \frac{1}{2} |u|_H^2 \middle| \mathcal{F}_\tau \right]$$

is an easy consequence of proposition 9. Let θ_0 be the measure on \mathbb{W} defined by

$$d\theta_0 = \frac{e^{-f}}{\mathbb{E}_\nu [e^{-f} | \mathcal{F}_\tau]} d\nu$$

According to corollary 3, there exists $(u_n) \in (\mathcal{D}_\tau^i)^\mathbb{N}$ such that ν -a.s.

$$\begin{aligned} \mathbb{E}_\nu \left[\frac{dW^{u_n} \nu}{d\nu} \log \frac{dW^{u_n} \nu}{d\nu} \middle| \mathcal{F}_\tau \right] &\rightarrow \mathbb{E}_\nu \left[\frac{d\theta_0}{d\nu} \log \frac{d\theta_0}{d\nu} \middle| \mathcal{F}_\tau \right] \\ \mathbb{E}_\nu \left[\frac{dW^{u_n} \nu}{d\nu} \log \frac{d\theta_0}{d\nu} \middle| \mathcal{F}_\tau \right] &\rightarrow \mathbb{E}_\nu \left[\frac{d\theta_0}{d\nu} \log \frac{d\theta_0}{d\nu} \middle| \mathcal{F}_\tau \right] \end{aligned}$$

Denote $L_n = \frac{dW^{u_n} \nu}{d\nu}$, since W^{u_n} is ν -a.s. invertible, we have

$$\mathbb{E}_\nu \left[f \circ W^{u_n} + \frac{1}{2} |u_n|_H^2 \middle| \mathcal{F}_\tau \right] = \mathbb{E}_\nu [L_n | \mathcal{F}_\tau] + \mathbb{E}_\nu [L_n \log L_n | \mathcal{F}_\tau]$$

When n goes to infinity, we have ν -a.s.

$$\mathbb{E}_\nu [L_n \log L_n | \mathcal{F}_\tau] \rightarrow \mathbb{E}_\nu \left[\frac{d\theta_0}{d\nu} \log \frac{d\theta_0}{d\nu} \middle| \mathcal{F}_\tau \right]$$

and since $f = -\log \frac{d\theta_0}{d\nu} - \log \mathbb{E}_\nu [e^{-f} | \mathcal{F}_\tau]$, ν -a.s.

$$\mathbb{E}_\nu [f L_n | \mathcal{F}_\tau] \rightarrow \mathbb{E}_\nu \left[f \frac{d\theta_0}{d\nu} \middle| \mathcal{F}_\tau \right]$$

So finally, when n goes to infinity, ν -a.s.

$$\begin{aligned} \mathbb{E}_\nu \left[f \circ W^{u_n} + \frac{1}{2} |u_n|_H^2 \middle| \mathcal{F}_\tau \right] &\rightarrow \mathbb{E}_{\theta_0} [f] + \mathbb{E}_\nu \left[\frac{d\theta_0}{d\nu} \log \frac{d\theta_0}{d\nu} \middle| \mathcal{F}_\tau \right] \\ &= -\log \mathbb{E}_\nu [e^{-f} | \mathcal{F}_\tau] \end{aligned}$$

which conclude the proof. \square

Theorem 6. *Set $f : \mathbb{W} \rightarrow \mathbb{R}$ a measurable function verifying $\mathbb{E}_\nu [|f|(1 + e^{-f})] < \infty$, then if there exists some $u \in \tilde{\mathcal{D}}_\tau \cap L_a^2(\nu, H)$ such that W^u is ν -a.s. left-invertible and $\frac{dW^u \nu}{d\nu} = \frac{e^{-f}}{\mathbb{E}_\nu [e^{-f} | \mathcal{F}_\tau]}$, then we have*

$$-\log \mathbb{E}_\nu [e^{-f} | \mathcal{F}_\tau] = \inf_{u \in \tilde{\mathcal{D}}_\tau \cap L_a^2(\nu, H)} \mathbb{E}_\nu \left[f \circ W^u + \frac{1}{2} |u|_H^2 \middle| \mathcal{F}_\tau \right] \quad \nu - a.s.$$

Proof: Since W^u is ν -a.s. left invertible and that $\frac{dW^u\nu}{d\nu} = \frac{e^{-f}}{\mathbb{E}_\nu[e^{-f}|\mathcal{F}_\tau]}$. We have

$$\frac{1}{2}\mathbb{E}_\nu[|u|_H^2|\mathcal{F}_\tau] = \mathbb{E}_\nu\left[\frac{e^{-f}}{\mathbb{E}_\nu[e^{-f}|\mathcal{F}_\tau]}\log\left(\frac{e^{-f}}{\mathbb{E}_\nu[e^{-f}|\mathcal{F}_\tau]}\right)\middle|\mathcal{F}_\tau\right]$$

and

$$\begin{aligned}\mathbb{E}_\nu\left[f \circ W^u + \frac{1}{2}|u|_H^2\middle|\mathcal{F}_\tau\right] &= \mathbb{E}_\nu\left[\frac{e^{-f}}{\mathbb{E}_\nu[e^{-f}|\mathcal{F}_\tau]}f + \frac{e^{-f}}{\mathbb{E}_\nu[e^{-f}|\mathcal{F}_\tau]}\log\left(\frac{e^{-f}}{\mathbb{E}_\nu[e^{-f}|\mathcal{F}_\tau]}\right)\middle|\mathcal{F}_\tau\right] \\ &= -\log\mathbb{E}_\nu[e^{-f}|\mathcal{F}_\tau]\end{aligned}$$

and we conclude the proof with proposition 9. \square

Theorem 7. Set $f : \mathbb{W} \rightarrow \mathbb{R}$ a measurable function such that

$$-\log\mathbb{E}_\nu[e^{-f}|\mathcal{F}_\tau] = \inf_{u \in \tilde{\mathcal{D}}_\tau \cap L_a^2(\nu, H)} \mathbb{E}_\nu\left[f \circ W^u + \frac{1}{2}|u|_H^2\middle|\mathcal{F}_\tau\right] \quad \nu - a.s.$$

Denote this infimum J_* . It is attained at $u \in \tilde{\mathcal{D}}_\tau \cap L_a^2(\nu, H)$ if and only if W^u is ν -a.s. left-invertible and $\frac{dW^u\nu}{d\nu} = \frac{e^{-f}}{\mathbb{E}_\nu[e^{-f}|\mathcal{F}_\tau]}$.

Proof: Denote $L = \frac{dW^u\nu}{d\nu}$. The direct implication is given by last theorem. Conversely, if W^u is not ν -a.s. left-invertible, $\mathbb{E}_\nu[L \log L|\mathcal{F}_\tau] < \frac{1}{2}\mathbb{E}_\nu[|u|_H^2|\mathcal{F}_\tau]$ and

$$\begin{aligned}-\log\mathbb{E}_\nu[e^{-f}|\mathcal{F}_\tau] &\leq \inf_{\alpha \in \tilde{\mathcal{D}}_\tau \cap L_a^2(\nu, H)} \mathbb{E}_\nu\left[f \circ W^\alpha + \frac{dW^\alpha\nu}{d\nu}\log\frac{dW^\alpha\nu}{d\nu}\middle|\mathcal{F}_\tau\right] \\ &\leq \mathbb{E}_\nu[f \circ W^u + L \log L|\mathcal{F}_\tau] \\ &< \mathbb{E}_\nu\left[f \circ W^u + \frac{1}{2}|u|_H^2\middle|\mathcal{F}_\tau\right]\end{aligned}$$

which is a contradiction.

We get $L = \frac{e^{-f}}{\mathbb{E}_\nu[e^{-f}|\mathcal{F}_\tau]}$ by uniqueness of the minimizing measure of $\inf_{\theta \in \mathcal{P}_\tau} \mathbb{E}_\theta[f + \log \frac{d\theta}{d\nu}|\mathcal{F}_\tau]$. \square

7. Prékopa-Leindler theorem for conditional expectations

Definition 6. We denote

$$H_b = \left\{h \in H, \dot{h} \text{ is } dt - a.s. \text{ bounded}\right\}$$

Remark: Observe that $H_b \subset \mathcal{D}$ and that if $u \in \mathcal{D}$, $u(w) \in H_b$ ν -a.s.

Theorem 8. Assume that for any $u \in \mathcal{D}$,

$$W^u(w) = W^{u(w)}(w) \quad \nu - a.s.$$

Set $t \in [0, 1]$. Set $a, b, c : \mathbb{W} \rightarrow \mathbb{R}$ positive and measurable such that for every $h, k \in H$ and $s \in [0, 1]$ we have ν -a.s.

$$a \circ W^{sh+(1-s)k} \exp\left(-\frac{1}{2}|sh + (1-s)k|_H^2\right) \geq \left(b \circ W^h \exp\left(-\frac{1}{2}|h|_H^2\right)\right)^s \left(c \circ W^k \exp\left(-\frac{1}{2}|k|_H^2\right)\right)^{1-s}$$

then for any density d such that $h \in H_b \mapsto -\log d \circ W^h$ is ν -a.s. concave and $\mathbb{E}_\nu[d|\mathcal{F}_\tau] = 1$, if θ denotes the measure on W given by $\frac{d\theta}{d\nu} = d$, we have in $\bar{\mathbb{R}}$:

$$\mathbb{E}_\theta[a|\mathcal{F}_\tau] \geq (\mathbb{E}_\theta[b|\mathcal{F}_\tau])^s (\mathbb{E}_\theta[c|\mathcal{F}_\tau])^{1-s}$$

Proof: First observe that eventually replacing a, b, c with da, db, dc and using Bayes formula we only need to prove the case $d = 1$ i.e. $\theta = \nu$

With the convention $\log(\infty) = \infty$ and $\log(0) = -\infty$, we denote

$$\tilde{a} = -\log a, \tilde{b} = -\log b, \tilde{c} = -\log c$$

We begin with the case where there exists $m, M > 0$ such that we have ν -a.s.

$$m \leq \tilde{a}, \tilde{b}, \tilde{c} \leq M$$

Set $s \in [0, 1]$ for $h, k \in H$, we have

$$\begin{aligned} & a \circ W^{sh+(1-s)k} \exp\left(-\frac{1}{2}|sh+(1-s)k|_H^2\right) \\ & \geq \left(b \circ W^h \exp\left(-\frac{1}{2}|h|_H^2\right)\right)^s \left(c \circ W^k \exp\left(-\frac{1}{2}|k|_H^2\right)\right)^{1-s} \end{aligned}$$

So for $u_1, u_2 \in \mathcal{D}_\tau^i$

$$\begin{aligned} & a \circ W^{su_1+(1-s)u_2} \exp\left(-\frac{1}{2}|su_1+(1-s)u_2|_H^2\right) \\ & \geq \left(b \circ W^{u_1} \exp\left(-\frac{1}{2}|u_1|_H^2\right)\right)^s \left(c \circ W^{u_2} \exp\left(-\frac{1}{2}|u_2|_H^2\right)\right)^{1-s} \end{aligned}$$

hence applying the logarithm function, changing the sign and taking the conditional expectation relative to \mathcal{F}_τ we obtain

$$\begin{aligned} & \mathbb{E}_\nu \left[\tilde{a} \circ W^{su_1+(1-s)u_2} + \frac{1}{2}|su_1+(1-s)u_2|_H^2 \middle| \mathcal{F}_\tau \right] \\ & \leq s \mathbb{E}_\nu \left[\tilde{b} \circ W^{u_1} + \frac{1}{2}|u_1|_H^2 \middle| \mathcal{F}_\tau \right] + (1-s) \mathbb{E}_\nu \left[\tilde{c} \circ W^{u_2} + \frac{1}{2}|u_2|_H^2 \middle| \mathcal{F}_\tau \right] \end{aligned}$$

So

$$\begin{aligned} & \inf_{u \in \mathcal{D}_\tau^i} \mathbb{E}_\nu \left[\tilde{a} \circ W^u + \frac{1}{2}|u|_H^2 \middle| \mathcal{F}_\tau \right] \\ & \leq s \mathbb{E}_\nu \left[\tilde{b} \circ W^{u_1} + \frac{1}{2}|u_1|_H^2 \middle| \mathcal{F}_\tau \right] + (1-s) \mathbb{E}_\nu \left[\tilde{c} \circ W^{u_2} + \frac{1}{2}|u_2|_H^2 \middle| \mathcal{F}_\tau \right] \end{aligned}$$

According to theorem 5 we have

$$-\log \mathbb{E}_\nu [e^{-\tilde{a}} | \mathcal{F}_\tau] \leq s \mathbb{E}_\nu \left[\tilde{b} \circ W^{u_1} + \frac{1}{2}|u_1|_H^2 \middle| \mathcal{F}_\tau \right] + (1-s) \mathbb{E}_\nu \left[\tilde{c} \circ W^{u_2} + \frac{1}{2}|u_2|_H^2 \middle| \mathcal{F}_\tau \right]$$

which implies

$$\begin{aligned} -\log \mathbb{E}_\nu [e^{-\tilde{a}} | \mathcal{F}_\tau] & \leq s \mathbb{E}_\nu \left[\tilde{b} \circ W^{u_1} + \frac{1}{2}|u_1|_H^2 \middle| \mathcal{F}_\tau \right] \\ & \quad + (1-s) \inf_{v \in \mathcal{D}_\tau^i} \mathbb{E}_\nu \left[\tilde{c} \circ W^v + \frac{1}{2}|v|_H^2 \middle| \mathcal{F}_\tau \right] \\ & = s \mathbb{E}_\nu \left[\tilde{b} \circ W^{u_1} + \frac{1}{2}|u_1|_H^2 \middle| \mathcal{F}_\tau \right] - (1-s) \log \mathbb{E}_\nu [e^{-\tilde{c}} | \mathcal{F}_\tau] \end{aligned}$$

which implies once again

$$\begin{aligned} -\log \mathbb{E}_\nu [e^{-\tilde{a}} | \mathcal{F}_\tau] &\leq s \inf_{v \in \mathcal{D}_\tau^i} \mathbb{E}_\nu \left[\tilde{b} \circ W^v + \frac{1}{2} |v|_H^2 \middle| \mathcal{F}_\tau \right] \\ &\quad - (1-s) \log \mathbb{E}_\nu [e^{-\tilde{c}} | \mathcal{F}_\tau] \\ &= -s \log \mathbb{E}_\nu [e^{-\tilde{b}} | \mathcal{F}_\tau] - (1-s) \log \mathbb{E}_\nu [e^{-\tilde{c}} | \mathcal{F}_\tau] \end{aligned}$$

taking the opposite and applying the exponential, we get

$$\mathbb{E}_\nu [e^{-\tilde{a}} | \mathcal{F}_\tau] \geq \left(\mathbb{E}_\nu [e^{-\tilde{b}} | \mathcal{F}_\tau] \right)^s \left(\mathbb{E}_\nu [e^{-\tilde{c}} | \mathcal{F}_\tau] \right)^{1-s}$$

For the general case, denote for $n \in \mathbb{N}$ and $m \in \mathbb{N}^*$

$$\begin{aligned} \tilde{a}_n &= \tilde{a} \wedge n, \tilde{b}_n = \tilde{b} \wedge n, \tilde{c}_n = \tilde{c} \wedge n \\ \tilde{a}_{nm} &= \tilde{a}_n + \frac{1}{m}, \tilde{b}_{nm} = \tilde{b}_n + \frac{1}{m}, \tilde{c}_{nm} = \tilde{c}_n + \frac{1}{m} \end{aligned}$$

For every $h, k \in H$, we have ν -a.s.:

$$\tilde{a}_{nm} \circ W^{sh+(1-s)k} + \frac{1}{2} |sh + (1-s)k|_H^2 \leq \tilde{b}_{nm} \circ W^h + \frac{1}{2} |h|_H^2 + (1-s) \tilde{c}_{nm} \circ W^k + \frac{1}{2} |k|_H^2$$

so the bounded case we treated above ensures that

$$\mathbb{E}_\nu [e^{-\tilde{a}_{nm}} | \mathcal{F}_\tau] \geq \left(\mathbb{E}_\nu [e^{-\tilde{b}_{nm}} | \mathcal{F}_\tau] \right)^s \left(\mathbb{E}_\nu [e^{-\tilde{c}_{nm}} | \mathcal{F}_\tau] \right)^{1-s}$$

The monotone limit theorem enables us to take the limit with relation to m and then to take it again with respect to n to get the result. \square

8. Examples

In this section we discuss several examples that fit into the framework we elaborated. Each time, we prove that the conditions of section 2 and definition 1 are satisfied, which ensure that every result from section 2 to 7 apply. We also discuss whether theorem 8 applies or not. See [9] for the omitted proofs concerning the diffusion, [10] for the omitted proofs concerning the other examples.

8.1. Diffusion. Set $m \leq d \in \mathbb{N}^*$ such that $m+d = n$, $c \in \mathbb{R}$, $\sigma : \mathbb{R}^m \rightarrow \mathcal{M}_{m,d}(\mathbb{R})$ and $b : \mathbb{R}^m \rightarrow \mathbb{R}^m$ bounded and lipschitz functions. σ_i will denote the i -th column of σ . Notice that every matrix will be identified with its canonical linear operator. Set $(\Omega, \theta, (\mathcal{G}_t))$ a probability space, V a θ -Brownian motion on Ω with values in \mathbb{R}^d . Set Y a \mathbb{R}^m -valued strong solution of the stochastic differential equation:

$$Y(t) = c + \int_0^t \sigma(Y(s)) dV(s) + \int_0^t b(Y(s)) ds$$

on $(\Omega, \theta, (\mathcal{G}_t), B)$. The hypothesis on σ and b ensure the existence and uniqueness of Y if we impose its paths to be continuous.

We denote μ the Wiener measure on $C([0, 1], \mathbb{R}^d)$ and μ^X the measure on $C([0, 1], \mathbb{R}^m)$ the image measure of Y .

We define the processes X and B on \mathbb{W} by:

$$\begin{aligned} X(t) &: (w, w') \in \mathbb{W} \mapsto w(t) \in \mathbb{R}^m \\ B(t) &: (w, w') \in \mathbb{W} \mapsto w'(t) \in \mathbb{R}^d \end{aligned}$$

Proposition 10. *Under $\mu^X \times \mu$, the law of X is μ^X , B is a Brownian motion and they are independent. There exists θ, η such that if we define $\beta_{\mathbb{X}}$ as*

$$\beta_{\mathbb{X}} = \int_0^\cdot \theta(X(s))dM(s) + \int_0^\cdot \eta(X(s))dB(s)$$

$\beta_{\mathbb{X}}$ is a $\mu^X \times \mu$ -Brownian motion and $\mu^X \times \mu$ -a.s.

$$X = c + \int_0^\cdot \sigma(X(s))d\beta_{\mathbb{X}}(s) + \int_0^\cdot b(X(s))ds$$

This construction of $\beta_{\mathbb{X}}$ is taken from [15].

Definition 7. *We denote*

$$\mathbb{X} = (X, \beta_{\mathbb{X}})$$

and $\mu^{\mathbb{X}}$ its image measure.

X is a $\mu^{\mathbb{X}}$ path-continuous strong solution of the stochastic differential equation

$$X = c + \int_0^\cdot \sigma(X(s))d\beta_{\mathbb{X}}(s) + \int_0^\cdot b(X(s))ds$$

For $u \in G_0(\mu^{\mathbb{X}}, \beta_X)$, set $\beta_{\mathbb{X}}^u = \beta + u$ and X^u the $\mu^{\mathbb{X}}$ -a.s. path-continuous strong solution of the stochastic differential equation

$$X^u = c + \int_0^\cdot \sigma(X^u(s))d\beta_{\mathbb{X}}^u(s) + \int_0^\cdot b(X^u(s))ds$$

Finally, we denote

$$\mathbb{X}^u = (X^u, \beta_{\mathbb{X}} + u)$$

Theorem 9. *$(\mathbb{W}, \mu^{\mathbb{X}}, \beta_{\mathbb{X}}, (\mathbb{X}^u)_{u \in \mathcal{D}})$ verify the conditions of section 2. $(\mathbb{W}, \mu_a, \beta_{\mathbb{X}}, (\mathbb{X}^u)_{u \in G_0(\mu^{\mathbb{X}}, \beta_{\mathbb{X}})})$ verify the conditions of definition 1.*

Proof: (vii) of definition 1 is clear, see [9] for the remainder of the proof. \square

Corollary 2. *It is clear that for every $u \in \mathcal{D}$, we clearly have $\mu^{\mathbb{X}}$ -a.s.*

$$\mathbb{X}^u(w) = \mathbb{X}^{u(w)}(w)$$

so theorem 8 applies.

8.2. Brownian bridge. We still denote μ the Wiener measure on \mathbb{W} . Set $a \in \mathbb{R}^n$, we denote μ_a the measure on \mathbb{W} such that for any bounded measurable function f we have

$$\mathbb{E}_{\mu_a}[f] = \mathbb{E}_{\mu}[f|W_1 = a]$$

μ_a can also be defined as follow : let \mathcal{E}_a be the Dirac measure in a , $\mathcal{E}_a(W_1)$ is a positive Wiener distributions hence it defines a Radon measure ν_a on \mathbb{W} , then

$$\mu_a = \left(\frac{1}{2\pi}\right)^n \nu_a$$

We recall the definition of a Brownian bridge:

Definition 8. *Set (Ω, \mathcal{G}, Q) a probability space. An a -Brownian bridge X under a probability Q is a path-continuous Gaussian process such that $\mathbb{E}_Q[X(t)] = at$ and $\text{cov}(X(s), X(t)) = ((s \wedge t) - st)I_d$*

Proposition 11. *W is an a -Brownian bridge under μ_a , and the process β_a defined as*

$$\beta_a(t) = W(t) - at + \int_0^t \frac{W(s) - as}{1-s} ds$$

is a Brownian motion under μ_a and the filtrations of β_a and W completed with respect to μ_a are equal. Moreover, we have

$$W(t) = at + (1-t) \int_0^t \frac{d\beta_a(s)}{1-s}$$

The following remark will be useful in next section.

Remark: For $a \in \mathbb{R}^n$ and $t \in [0, 1]$, we have μ_a -a.s.

$$\beta_a(t) = W_t + \int_0^t \frac{W_s - a}{1-s} ds$$

Definition 9. For $u \in G_0(\mu_a, \beta_a)$, we denote $\beta_a^u = \beta_a + u$.

Proposition 12. Set $u \in G_0(\mu_a, \beta_a)$, then there exists a unique μ_a -a.s. path continuous process W_a^u such that

$$W_a^u(t) = \beta_a^u(t) + at - \int_0^t \frac{W_a^u(s)}{1-s} ds$$

Furthermore, we have

$$\begin{aligned} W_a^u(t) &= at + (1-t) \int_0^t \frac{d\beta_a^u(s)}{1-s} \\ &= W(t) + \int_0^t \left(\dot{u}(s) - \int_0^s \frac{\dot{u}(r)}{1-r} dr \right) ds \end{aligned}$$

Theorem 10. $(\mathbb{W}, \mu_a, \beta_a, (W_a^u)_{u \in \mathcal{D}})$ verify the conditions of section 2. $(\mathbb{W}, \mu_a, \beta_a, (W_a^u)_{u \in G_0(\mu_a, \beta_a)})$ verify the conditions of definition 1.

Proof: (vii) of definition 1 is clear, see [10] for the remainder of the proof. □

Corollary 3. It is clear that for every $u \in \mathcal{D}$, we clearly have μ_a -a.s.

$$W_a^u(w) = W_a^{u(w)}(w)$$

so theorem 8 applies.

8.3. Loop measure. We keep the notations of last section. Denote

$$S = \{a \in \mathbb{R}^n, |a| = 1\}$$

and set $\alpha : S \rightarrow \mathbb{R}_+$ a locally lipschitz function such that $\{x, \alpha(x) \neq 0\}$ is of strictly positive measure for the Lebesgue measure on S and

$$\int_S \alpha(a) da = 1$$

We define the measure ν_l as follow: for any bounded measurable function f on W , we set

$$\mathbb{E}_{\nu_l}[f] = \int_S \alpha(a) \mathbb{E}_{\mu_a}[f] da$$

For more on loop measures, see Fang's work in [6].

Definition 10. We denote

$$\begin{aligned} h_a : (t, x) &\in [0, 1) \times \mathbb{R}^n \mapsto \left(\frac{1}{\pi(1-t)} \right)^{\frac{n}{2}} \exp \left(\frac{-|x-a|^2}{2(1-t)} \right) \\ h : (t, x) &\in [0, 1) \times \mathbb{R}^n \mapsto \int_S \alpha(a) h_a(t, x) da \end{aligned}$$

Proposition 13. Set $a \in \mathbb{R}^n$ and $t \in [0, 1)$, then

$$\left. \frac{d\mu_a}{d\mu} \right|_{\mathcal{F}_t^W} = h_a(t, W_t)$$

Proposition 14. Set $t \in [0, 1)$, we have

$$\left. \frac{d\nu}{d\mu} \right|_{\mathcal{F}_t^W} = h(t, W_t)$$

Proposition 15. Define

$$\beta_{lp}(t) = W(t) - \int_0^t \frac{h'(s, W(s))}{h(s, W(s))} ds$$

where h' designates the partial derivative of h with respect to x .

Then β_{lp} is a ν_l Brownian motion and the filtrations of W and β_{lp} completed with respect to ν_l are equal.

Definition 11. For $u \in G_0(\nu_l, \beta_{lo})$, we denote $\beta_{lp}^u = \beta_{lp} + u$.

Proposition 16. Set $u \in G_0(\nu_l, \beta_{lo})$, then there exists a unique ν_l -a.s. path continuous process W_{lp}^u such that

$$W_{lp}^u(t) = \beta_{lp}^u(t) + \int_0^t \frac{h'(s, W_{lo}^u(s))}{h(s, W_{lo}^u(s))} ds$$

Theorem 11. $(\mathbb{W}, \nu_l, \beta_{lp}, (W_{lp}^u)_{u \in \mathcal{D}})$ verify the conditions of section 2. $(\mathbb{W}, \nu_l, \beta_{lo}, (W_{lp}^u)_{u \in G_0(\nu_l, \beta_{lo})})$ verify the conditions of definition 1.

Proof: (vii) of definition 1 is clear, see [10] for the remainder of the proof. □

Corollary 4. It is clear that for every $u \in \mathcal{D}$, we clearly have ν_l -a.s.

$$W_{lo}^u(w) = W_{lo}^{u(w)}(w)$$

so theorem 8 applies.

8.4. Diffusing particles without collision. Set $\sigma, b, \delta, \gamma \in \mathbb{R}$ such that

$$\sigma^2 \leq 2\gamma$$

The proof of the following theorem can be found in [16] or [3].

Theorem 12. *Set $(\Omega, \theta, (\mathcal{G}_t))$ a filtered probability space, $(z_1(0), \dots, z_n(0)) \in \mathbb{R}^n$ and $B = (B_1, \dots, B_n)$ a \mathbb{R}^n -valued θ -Brownian motion. We consider the following stochastic differential system:*

$$\begin{aligned} Z_1(t) &= z_1(0) + \sigma B_1(t) + b \int_0^t Z_1(s) ds + ct + \gamma \sum_{j \in \{1, \dots, n\} \setminus \{1\}} \int_0^t \frac{ds}{Z_1(s) - Z_j(s)} \\ &\vdots \\ Z_n(t) &= z_n(0) + \sigma B_n(t) + b \int_0^t Z_n(s) ds + ct + \gamma \sum_{j \in \{1, \dots, n\} \setminus \{n\}} \int_0^t \frac{ds}{Z_n(s) - Z_j(s)} \end{aligned}$$

under the condition that θ -a.s. for every $t \in [0, \infty)$

$$Z_1(t) \leq \dots \leq Z_n(t)$$

This system admits a unique strong solution on $(\Omega, \theta, (\mathcal{G}_t), B)$ and the first collision time is θ -a.s. equal to ∞ .

Consider $(\Omega, \theta, (\mathcal{G}_t))$ a filtered probability space, $(z_1(0), \dots, z_n(0)) \in \mathbb{R}^n$ and $B = (B_1, \dots, B_n)$ a \mathbb{R}^n -valued θ -Brownian motion, and Z the strong solution of the stochastic differential system of theorem 12. Denote $\nu_{pa} = Z$ the image measure of Z . For $1 \leq i \leq n$, denote W_1, \dots, W_n the coordinates of W and define

$$M_i(t) = W_i(t) - z_i(0) - b \int_0^t W_i(s) ds - ct - \gamma \sum_{j \in \{1, \dots, n\} \setminus \{i\}} \int_0^t \frac{ds}{W_i(s) - W_j(s)}$$

and

$$M = (M_1, \dots, M_n)$$

M is a local martingale and

$$\langle M_i, M_j \rangle(t) = \sigma^2 t$$

Define

$$\beta_{pa} = \frac{1}{\sigma} M$$

Levy theorem clearly ensures that β is a ν_{pa} -Brownian motion and we clearly have for every $1 \leq i \leq n$,

$$W_i(t) = z_i(0) + \sigma \beta_{pa,i}(t) + b \int_0^t W_i(s) ds + ct + \gamma \sum_{j \in \{1, \dots, n\} \setminus \{i\}} \int_0^t \frac{ds}{W_i(s) - W_j(s)}$$

For $u \in G_0(\nu_{pa}, \beta_{pa})$ denote

$$\beta_{pa}^u = \beta_{pa} + u$$

and ν_{pa}^u the probability measure given by

$$\frac{d\nu_{pa}^u}{d\nu_{pa}} = \rho(-\delta_{\beta_{pa}} u)$$

According to Girsanov theorem, $\beta_{pa} + u$ is a Brownian motion under ν_{pa}^u , so according to theorem 12, there exists a unique ν_{pa}^u -a.s. continuous process $W_{pa}^u = (W_{pa,i}^u, \dots, W_{pa,n}^u)$ such that ν_{pa}^u -a.s. for every $1 \leq i \leq n$

$$W_{pa,i}^u(t) = z_i(0) + \sigma \beta_{pa,i}^u(t) + b \int_0^t W_{pa,i}^u(s) ds + ct + \gamma \sum_{j \in \{1, \dots, n\} \setminus \{i\}} \int_0^t \frac{ds}{W_{pa,i}^u(s) - W_{pa,j}^u(s)}$$

and ν_{pa}^u -a.s. for every $t \in [0, 1]$

$$W_{pa,1}^u(t) \leq \dots \leq W_{pa,n}^u(t)$$

Since $\nu_{pa}^u \sim \nu_{pa}$, W^u is ν_{pa} -a.s. continuous and ν_{pa} -a.s. for every $1 \leq i \leq n$

$$W_{pa,i}^u(t) = z_i(0) + \sigma \beta_{pa,i}^u(t) + b \int_0^t W_{pa,i}^u(s) ds + ct + \gamma \sum_{j \in \{1, \dots, n\} \setminus \{i\}} \int_0^t \frac{ds}{W_{pa,i}^u(s) - W_{pa,j}^u(s)}$$

and ν_{pa} -a.s. for every $t \in [0, 1]$

$$W_{pa,1}^u(t) \leq \dots \leq W_{pa,n}^u(t)$$

Theorem 13. $(\mathbb{W}, \nu_{pa}, \beta_{pa}, (W_{pa}^u)_{u \in \mathcal{D}})$ verify the conditions of section 2.

$(\mathbb{W}, \nu_{pa}, \beta_{pa}, (W_{pa}^u)_{u \in G_0(\nu_{pa}, \beta_{pa})})$ verify the conditions of definition 1.

Proof: (vii) of definition 1 is clear, see [10] for the remainder of the proof. □

Corollary 5. It is clear that for every $u \in \mathcal{D}$, we clearly have ν_{pa} -a.s.

$$W_{pa}^u(w) = W_{pa}^{u(w)}(w)$$

so theorem 8 applies.

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